Convergence and stability of a semi-Lagrangian method for Burgers' equation with Dirichlet boundary conditions

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Abstract

It is well known that the semi-Lagrangian method is used to simulate advection-type flow problems in computational fluid dynamics (CFD) (see, for example, [1, 2, 3, 4, 6, 7, 8, 9], etc.). The movement of particles is described in a flow field along their characteristic curves as a solution of nonlinear Cauchy problem in a fixed framework for each discretized time unit. One of the reasons contributing to the popularity of this method is that it can avoid bad representation problems of fluid features occurring in the Lagrangian method such as the accumulation of particles along characteristic curves or vacancy in a spatial domain for a long-time simulation [5]. Especially, it is known that the backward semi-Lagrangian method (BSLM) has a good stability property such that it is free from the Courant-Friedrichs-Lewy (CFL) condition between the temporal step size and the spatial grid size. In this presentation, we give a talk about a concrete convergence analysis with stability for a backward semi-Lagrangian method in a non-linear Burgers' equation with the Dirichlet boundary conditions. The material time derivative and the diffusion term along characteristic curves are discretized by backward difference formula of type 2 (BDF2) and the second order central finite difference respectively, together with the local Lagrangian interpolation. The Cauchy problem for characteristic curves is resolved by an iteration-free method, error correction method based on a modified quadratic polygon. Using mathematical induction hypotheses under the mesh restriction $\Delta t = O(\Delta x^{1/2})$ between the temporal step size Δt and the spatial grid size Δx, we prove that the proposed method has the convergence order $O(\Delta t^2 + \Delta x^2 + \Delta x^p/\Delta t)$ in the sense of the discrete H_1-norm, where p is the degree of an interpolation polynomial. Further, we establish the unconditional stability of the method and present numerical tests to support the theoretical analyses.

Keywords: Burgers' equation; Semi-Lagrangian method; Non-linear advection–diffusion equation; Convergence analysis; Stability analysis.

References

[1] Allievi, A. and Bermejo, R. (2000). Finite element modified method of characteristics for the Navier–Stokes equations. Int. J. Numer. Methods Fluids, 32(4):439–463.

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[2] Batchelor, G. K. (2000). An Introduction to Fluid Dynamics. Cambridge Mathematical Library. Cambridge University Press.

[3] Bermudez, A., Nogueiras, M., and Vazquez, C. (2006a). Numerical analysis of convection–diffusion–reaction problems with higher order characteristics/finite elements. part I: time discretization. SIAM J. Numer. Anal., 44(5):1829–1853.

[4] Bermudez, A., Nogueiras, M., and Vazquez, C. (2006b). Numerical analysis of convection–diffusion–reaction problems with higher order characteristics/finite elements. part II: fully discretized scheme and quadrature formulas. SIAM J. Numer. Anal., 44(5):1854–1876.

[5] Ewing, R. E. and Wang, H. (2001). A summary of numerical methods for time-dependent advection–dominated partial differential equations. J. Comput. Appl. Math., 128(1):423–445.

[6] Landau, L. D. and Lifshitz, E. M. (1987). Fluid mechanics 2nd edition (course of theoretical physics), volume 6. Butterworth-Heinemann.

[7] Malevsky, A. V. and Thomas, S. J. (1997). Parallel algorithms for semi–Lagrangian advection. Int. J. Numer. Methods Fluids, 25(4):455–473.

[8] Robert, A. (1981). A stable numerical integration scheme for the primitive meteorological equations. Atmos. Ocean, 19(1):35–46.

[9] Staniforth, A. and Cote, J. (1991). Semi–Lagrangian integration schemes for atmospheric models–a review. Mon. Weather Rev., 119(9):2206–2223.